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# Venture Capital Financed Investments in Intellectual Capital

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## **Abstract**

The paper considers a new company in the IT industry, founded by a management team and partially financed by venture capital. Among the questions that the paper addresses are: how much venture capital should be acquired to help finance the development of the firm? How should a wish to grow, with the aim of making a breakthrough in the IT area, be balanced against the stockholders' wish to consume? The problem is studied as an optimal control problem with a random time horizon and we derive a series of prescriptions for investment and financial decisions.

*JEL classification:* D92;G24;O32

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optimal control.

## 1. Introduction

The information technology (IT) industry has been characterized by a rapid growth of the number of new firms entering the industry. New product and service innovations are being developed and marketed in increasing numbers. Growth has been fast in software development, e-business, internet services, and web design, but also the telecommunications industry has seen a rapid increase in new equipment and services. Many new firms have experienced a fast growth in terms of the number of employees, but the growth has necessarily not been financially sound. Some companies have used substantial parts of their equity, and the venture capital they attracted, to cover losses incurred by attempts to grow rapidly, and others headed for bankruptcy due to severe operating losses. Quite a few businesses were liquidated without having generated a single dollar of revenue.

This paper sets up a model of investment and financial decision making in a new firm, and derives a series of prescriptions for its growth and financing. Suppose that the company is founded at time  $t = 0$ , by acquiring another firm in the industry. The firm thus acquires a stock of intangible assets: specialized employee skills, knowledge, expertise, experience, and information. It has been argued that "we are entering the knowledge society in which the basic economic resource is no longer capital ... but is and

will be knowledge” (Drucker (1995)). Zingales (2000) posited that during the last decade, knowledge has replaced capital stocks as the main asset of a firm. Supposing knowledge can be capitalized, we summarize the firm’s intangible assets in a *stock of intellectual capital (IC)* (see also McAdam and McCreedy (1999)).

We consider a particular objective of the firm, being the development of a specific innovation (an IT product or service) which will make a *break-through* in its specific area. The date of the innovation,  $T$ , is unknown at time zero where the firm must make a plan for its investments in IC. In the time period until  $T$ , making investments in its stock of IC will increase the probability that the innovation is made by time  $T$ . At the initial instant of time, the firm, represented by a founding management team, can attract venture capital. The purpose of this is to increase the stock of IC that is acquired at the initial instant of time. The venture capitalist becomes a stockholder of the firm.

The paper deals with two main problems, essentially seen from the point of view of the founding management team. The first problem concerns how much, if any, venture capital that should be attracted initially. The second problem is how to design a strategy for the firm’s later investments in IC, in order to raise the probability of making the discovery.

In both areas, there is an extensive literature which we do not intend to survey here. The venture capital literature often is concerned with the contracting problem only, that is, how to regulate the relationship between

the entrepreneur and the venture capitalist. The problem can be seen as an agency problem; see, for instance, Sahlman (1988, 1990), Admati and Pfleiderer (1994), Kaplan and Strömberg (1993). Bergemann and Hege (1997) considers a problem with a closer relationship to our paper. Their paper deals with venture capital financing as well as investment decisions over time.

In the investment area, Schwartz and Zozaya-Gorostiza (2003) study a setup in which an organization undertakes an IT investment project, which has a random completion date. The project involves an option of spending initially an amount of money to acquire an IT asset. After the acquisition, the organization starts to receive a (random) cash flow and can invest in the development of the asset. The methodology used to evaluate the project is real options, using stochastic differential equations and Hamilton-Jacobi-Bellman equations. The financial aspects are, however, ignored.

Finally there is a stream of literature in the area of R&D that studies stochastic, dynamic investment problems. Examples are Reinganum (1981, 1982), Kamien and Schwartz (1982), Fudenberg et al. (1983), Grossman and Shapiro (1986), Doraszelski (2003). The idea here is to model the hazard rate of successful innovation as a function of the firm's current and/or cumulative investments.

The paper is organized as follows. Section 2 models the firm's situation as an optimal control problem with a random time horizon. Section 3 states our main results concerning optimal investment policies. All proofs

of the results in Section 3 are given in the appendix. Section 4 addresses the financial problem of how much venture capital to attract. Section 5 contains our concluding remarks.

## 2. Optimal Control Model

Time  $t$  is continuous and at time  $t = 0$  the firm starts its operations. To increase readability, we present the stochastic optimal control model in a series of subsections.

### 2.1. Accounting

At time zero, a founding management team contributes a fixed amount of capital  $X_0 \geq 0$  and the firm attracts venture capital in amount of  $L_0 \geq 0$ . It may happen that the firm is entirely financed by venture capital (cf. Bergemann and Hege (1997)), in which case we have  $X_0 = 0, L_0 > 0$ .

The venture capital is used as an initial, one-shot, investment in the stock of IC. The initial amount of available funds,  $X_0 + L_0$ , are spent to acquire a stock of intellectual capital,  $z_0 > 0$ . Let  $z(t)$  be a state variable that represents the dollar value of the firm's stock of IC by time  $t \geq 0$ . To simplify, suppose that the stock  $z(t)$  is the firm's only asset. The firm is likely to have relatively few tangible assets (machinery, equipment, materials), compared to the intangible ones, and we have chosen to disregard the tangible assets (cf. Brander and Lewis (1986)). Moreover, the cash balance

is zero at any instant of time.

## 2.2. *Financing*

We suppose that after having obtained the venture capital  $L_0$ , the firm can attract no more capital. Thus, the firm must be self-financed for  $t > 0$ . This approach to financing differs sharply from one in which a firm can attract external funds (equity and/or debt) at any instant of time (e.g., Schworm (1980), Bensoussan and Lesourne (1981)). On the other hand, Kamien and Schwartz (1982) argued that R&D investments, which are in many respects comparable to investments in IC, should be entirely self-financed.

It is also important to note that the assumed, one-shot venture capital financing differs from the common practice in which ventures are financed stagewise (see, e.g., Sahlman (1990), Kaplan and Strömberg (2003)). Thus, what we have in mind is a situation in which the venture capitalist provides only startup financing or one in which the capitalist provides all of the funding commitment on signing the contract. (The latter arrangement was used only in a minority (15%) of the cases examined in Kaplan and Strömberg (2003)).

Thus, having obtained the venture capital, the firm must finance any further investments in IC by retentions. The assumption here is that the firm allocates a constant part of its IC, say,  $\bar{z} > 0$ , to operations that are not directly connected to the efforts of making the breakthrough. These

operations generate revenue at the constant rate  $\pi > 0$ . Any excess intellectual capital,  $z(t) - \bar{z}$ , is devoted to trying to make the breakthrough, but this capital generates no operating revenue in the time interval  $[0, T]$ , where  $T$  is the date at which the breakthrough is made. This instant of time is a priori unknown. To save on notation, we define  $Z(t) = z(t) - \bar{z}$ , and impose the constraint  $Z(t) \geq 0$ .

The agreement between the firm and the venture capitalist is a contract according to which the venture capitalist gets security for her investment in the firm in the form of stocks (typically, convertible preferred stock). The venture capitalist thus gets stock in amount of  $L_0$ .

The stock holders (the venture capitalist and the founders) may receive returns on their investments, if so decided by the board. The payment of dividends, however, decreases the funds available for investments in IC.

With regard to the venture capitalist's exit, we suppose that the agreement states the capitalist exits when the breakthrough is made, that is, at time  $T$ , but she will exit at a predetermined instant of time,  $\tau = \text{const.} > 0$ , if the breakthrough has not yet been made by time  $\tau$ . Hence, since the innovation date is unknown, the exit time of the capitalist is random and given by  $t_e = \min\{T, \tau\}$ . This arrangement gives the capitalist the opportunity to leave the firm if the breakthrough has not been made within reasonable time. The exit of the capitalist at time  $\tau$  is a typical provision in a venture capital contract, and is quite similar to the required repayment of the principal at the maturity of a debt claim (Kaplan and Strömberg



(2003)). Section 2.4 discusses in detail how the venture capitalist is paid off when exiting the firm.

### 2.3. *Accumulation of Intellectual Capital*

Investments in IC are purposeful efforts of the firm (hiring new staff, training existing staff), made in order to increase the stock  $Z(t)$ . The single purpose of investing is to increase the chances of making the breakthrough. Let  $a(t) \geq 0$  denote the investment rate at time  $t$ . This is a control variable of the firm and is also the firm's only operating cost. The accumulation dynamics over the time interval  $[0, \infty)$  are given by

$$\dot{Z}(t) = a(t); \quad Z_0 > 0. \quad (1)$$

The absence of a depreciation term in (1), and nonnegativity of  $a(t) \geq 0$ , imply that the stock  $Z$  cannot decrease. Thus, our assumption is that the firm cannot decrease  $Z$  by laying off employees ( $a < 0$ ); neither does the stock  $Z$  decrease for the reason that knowledge becomes obsolete. To model organizational forgetting one can subtract a term  $\delta Z(t)$ ,  $\delta = \text{const.} > 0$ , on the right-hand side of (1). Then the firm's recent experiences would be more important for making the innovation than the distant experiences.

It can be argued that there should be a concave relationship in (1) between the investment rate and the rate of change of the stock, to reflect decreasing marginal effects of investment. We shall capture such a feature in another way (cf. (3)).

Consider the time at which a breakthrough occurs. This instant is represented by a random variable  $T$ , which is defined on a probability space  $(\Omega, \mathcal{F}, P_{a(\cdot)})$  and takes its values in  $[0, \infty)$ . The probability measure  $P$  depends on the control (although indirectly) in the following way:

$$P_{a(\cdot)}(T \in [t, t + dt) \mid T \geq t) = \gamma Z(t)dt + o(dt), \quad (2)$$

where  $o(dt)/dt \rightarrow 0$  uniformly in  $(Z, a)$  for  $dt \rightarrow 0$ . The term  $\gamma Z(t)$  on the right-hand side of (2) is the stopping rate and  $\gamma > 0$  is a constant. The stopping rate is increasing in  $Z$ , and (2) implies

$$\Pr(T \in [0, t)) = 1 - \exp \left[ -\gamma \int_0^t Z(s)ds \right]. \quad (3)$$

Equation (3) shows that the more intellectual capital that was accumulated during the time interval  $[0, t)$ , the higher the probability of making the breakthrough by time  $t$ . Thus, the firm's current chances of making the innovation depend on its stock of knowledge; knowledge has value. This hypothesis was employed by, e.g., Fudenberg et al. (1983) and Doraszelski (2003) and seems more plausible than the one used in other models of innovation (e.g., Reinganum (1981, 1982), Bergemann and Hege (1997), Dockner et al. (2000)). In these works, the probability of making a breakthrough depends on current investment only and hence is independent of accumulated knowledge. Then knowledge is irrelevant for the firm's current efforts.

Recall from (1) that current investments translate linearly into IC. However, due to (3), there is a diminishing marginal efficiency of IC in increasing

the stopping rate (since the probability in (3) grows in a concave fashion).

The specification in (2) also implies  $\Pr(T = \infty) = \exp \left[ -\gamma \int_0^\infty Z(s) ds \right]$ . Hence, a necessary and sufficient condition for having a finite completion time  $T$  is that  $\int_0^t Z(s) ds \rightarrow +\infty$  for  $t \rightarrow +\infty$ . This is true, as  $Z_0$  is positive and  $Z(\cdot)$  is nondecreasing.

#### 2.4. Objective Function

To construct the firm's objective function, we need to discuss the following issues. What happens when the breakthrough is made, and what will the venture capitalist receive, in return on her investment, upon her exit from the firm?

As to the first question, we summarize the firm's situation in a simple, but standard, way. Let  $\bar{P} = \text{const.} > 0$  denote the expected value of an uncertain reward to be earned at time  $T$ . (One can think of  $\bar{P}$  as being the value of a patent). The reward being constant means, in particular, that it does not depend on  $Z(T)$ , the stock of IC at the innovation date. Hence the reward is the same whether the innovation is made by a small or a large firm (cf. Grossman and Shapiro (1986)). As concerns the exit time  $t_e$  of the venture capitalist, we distinguish two cases:

*Case 1:*  $t_e = T \iff T < \tau$ .

The reward  $\bar{P}$  earned at time  $T$  is divided among the founders and the venture capitalist such that the latter gets  $\theta(L_0)\bar{P}$  and the former gets the

rest,  $(1-\theta(L_0))\bar{P}$ , where  $\theta(L_0) \in [0, 1]$ . The share  $\theta(L_0)$  depends only on the amount of venture capital provided; thus the share is time-invariable and does not depend on the innovation time  $T$ . We assume  $\theta(0) = 0, \theta'(L_0) > 0, \theta''(L_0) \neq 0$ .

*Case 2:*  $t_e = \tau \iff T > \tau$ .

The breakthrough has not yet been made by time  $\tau$ , which is the pre-determined, latest exit time of the venture capitalist. Hence she exits at  $t = \tau$  and receives, according to the redemption provision of the contract, an amount of  $\beta L_0$ ,  $\beta = \text{const.} > 0$ . If  $\beta = 1$ , the capitalist receives the principal amount of stock in the company. In the majority of venture capital contracts, however,  $\beta$  is greater than one. If the contract provides for accruing dividends, our assumption is that the cumulated amount is included in  $\beta L_0$  (which can be accomplished by a suitable choice of  $\beta$ ).

Denoting the firm's positive and constant rate of time preference by  $i$ , the objective functional,  $J$ , of the firm can now be constructed:

$$\begin{aligned} J(a(\cdot); Z_0) = & E_{a(\cdot)}^{T < \tau} \left[ \int_0^\tau e^{-it} [\pi - a(t)] dt + e^{-iT} (1 - \theta(L_0)) \bar{P} \right] + \\ & E_{a(\cdot)}^{T \geq \tau} \left[ \int_\tau^T e^{-it} [\pi - a(t)] dt + e^{-iT} \bar{P} - e^{-i\tau} \beta L_0 \right]. \quad (4) \end{aligned}$$

The objective is the maximization of the expected present value (at time zero) of the profit stream  $\pi - a(t)$  and the reward  $\bar{P}$ , minus the expected payment,  $\theta(L_0)\bar{P}$  or  $\beta L_0$ , upon the exit of the venture capitalist.

A pair  $(Z, a)$  is feasible if the objective value  $J$  is finite, state equation (1) is satisfied, and the control constraint  $a(t) \in [0, \pi]$  is fulfilled for  $t \in [0, \infty)$ .

### 3. Main Results

This section characterizes an optimal solution of the stochastic optimal control problem in Section 2, but without stating any proofs. These can be found in the appendix.

Using Hamilton-Jacobi-Bellman equations is a standard way to solve a stochastic optimal control problem. It turns out, however, that in the setup at hand this technique is less useful. The reason is that the problem is nonautonomous, leading to a partial differential equation for the value function. Instead, we apply a maximum principle for an infinite horizon, deterministic optimal control problem. This problem is equivalent to the stochastic optimal control problem with random stopping time that was stated in Section 2 (cf. Boukas et al. (1990), Carlson et al. (1991), Sorger (1991)).

The deterministic problem has two state variables,  $Z(t)$  and  $Y(t)$ , where the latter will be defined below. It turns out to be expedient to solve the problem backwards, and to divide it in two subproblems.

Given any feasible pair  $(Z_\tau, Y_\tau) = (Z(\tau), Y(\tau))$  that results from using an optimal investment policy on the interval  $[0, \tau)$ , we first solve a subprob-

lem on the time interval  $[\tau, \infty)$ . Denote this subproblem by  $P2$  and define the state variable

$$Y(t) = Y_\tau + \int_\tau^t \gamma Z(s) ds,$$

in which  $Y_\tau$  is a fixed but, so far, arbitrary real number. The objective functional of problem  $P2$ , denoted by  $J_2$ , represents the expected present value at time  $t = \tau$  and is given by

$$\begin{aligned} J_2(a(\cdot); Z_\tau, Y_\tau) = E_{a(\cdot)} \left[ \int_\tau^T e^{-i(t-\tau)} [\pi - a(t)] dt + e^{-i(T-\tau)} \bar{P} - \beta L_0 \right] = \\ - \beta L_0 + e^{i\tau + Y_\tau} \int_\tau^\infty e^{-[it + Y(t)]} [\pi - a(t) + \bar{P} \gamma Z(t)] dt, \quad (5) \end{aligned}$$

and the state dynamics are

$$\begin{aligned} \dot{Z}(t) &= a(t), \quad Z(\tau) = Z_\tau. \\ \dot{Y}(t) &= \gamma Z(t), \quad Y(\tau) = Y_\tau. \end{aligned} \quad (6)$$

When a solution of  $P2$  has been obtained, one can determine the optimal value of  $J_2$ , which will be denoted by  $J_2^*(Z_\tau, Y_\tau)$ .

Then we solve the full problem on the time interval  $[0, \infty)$ . This problem has the following objective functional, when cast as a deterministic optimal control problem:

$$J(a(\cdot), Z_0) = \int_0^\tau e^{-[it+Y(t)]} [\pi - a(t) + (1 - \theta(L_0)) \bar{P} \gamma Z(t)] dt + e^{-i\tau} J_2^*(Z_\tau, Y_\tau). \quad (7)$$

For  $t \in [0, \tau)$ , the dynamics of the problem are given by the state equations

$$\begin{aligned} \dot{Z}(t) &= a(t), & Z(0) &= Z_0 \\ \dot{Y}(t) &= \gamma Z(t), & Y(0) &= 0. \end{aligned} \quad (8)$$

The calculations will proceed as follows. Section 3.1 provides a solution of  $P2$  and Section 3.2 solves the full problem.

### 3.1. Solution of $P2$

Define costate variables  $\lambda_1(t), \lambda_2(t)$  associated with state variables  $Z$  and  $Y$ , respectively. In (5), the term  $e^{i\tau+Y_\tau}$  is constant and can be disregarded in the characterization of an optimal solution. The Hamiltonian is

$$H = e^{-(it+Y)} [\pi - a + \bar{P} \gamma Z] + \lambda_1 a + \lambda_2 \gamma Z,$$

which is linear in the control  $a$ , and hence an optimal investment policy is bang-bang or singular. There are three candidate policies. In terms of the

costate  $\lambda_1(t)$  they can be stated as

$$\begin{aligned}
 \textit{Dividend} \quad & : \quad a(t) = 0 \implies \lambda_1(t)e^{it+Y(t)} < 1 \\
 \textit{Investment} \quad & : \quad a(t) = \pi \implies \lambda_1(t)e^{it+Y(t)} > 1 \\
 \textit{Singular} \quad & : \quad a(t) \in [0, \pi] \implies \lambda_1(t)e^{it+Y(t)} = 1.
 \end{aligned} \tag{9}$$

The time function  $\lambda_1(t)e^{it+Y(t)}$  has an interpretation as a shadow price of the stock  $Z$ , which provides an intuition for the policies in (9). Notice that since stockholders are risk neutral, the value of a marginal dollar of net profit equals one. A *dividend policy* pays maximally to the stockholders and since there are no investments, the stock of IC remains constant. This happens if a unit of IC has a shadow price less than one and then the marginal dollar should be paid to the owners. Since the stock  $Z$  cannot be decreased (to get additional funds to pay as dividends), it is kept constant. Using an *investment policy*, the stock of intellectual capital grows at its maximal rate; there is no payout to stockholders. The reason is that the shadow price exceeds the value to the stockholders of having the marginal dollar paid out as dividends. Then the dollar is spent entirely on investments. Finally, if the shadow price equals one, stockholders are indifferent between investing the marginal dollar and having it paid out.

A particular interest relates to the stock level  $Z$  associated with a singular policy. Suppose that one chooses the singular control as  $a = 0$ . Then



the singular stock level,  $Z_S$ , say, is constant and equals

$$Z_S = \frac{1}{\gamma} \left( -i + \sqrt{\gamma(i\bar{P} - \pi)} \right). \quad (10)$$

To have a positive  $Z_S$ , we introduce the assumption

$$\bar{P} > \frac{\pi}{i} + \frac{i}{\gamma}. \quad (11)$$

Note that (11) implies  $\bar{P} > \frac{\pi}{i}$ . This inequality means that the expected reward exceeds the present value of a perpetuity of  $\pi$ . Otherwise the firm would have no incentive to try to make the innovation.

The instant of time at which the stock  $Z(t)$  reaches the singular level  $Z_S$ , when starting out in state  $Z_\tau$  and using the investment policy, is given by

$$t_S = \frac{Z_S - Z_\tau}{\pi}. \quad (12)$$

Proposition 1 characterizes an optimal solution of  $P2$ . The reader should be aware that the optimality conditions applied in the proposition are necessary only and require the existence of an optimal control.. Due to the fact that the discount factor involves the state variable  $Y$ , it is not possible to verify standard sufficiency conditions based upon concavity of Hamiltonians.

**Proposition 1** *For any  $Z_\tau \geq Z_0$ , an optimal solution of  $P2$  is:*

(i): *If  $Z_\tau \geq Z_S$  then  $a(t) = 0$  for  $t \in [\tau, \infty)$ .*

(ii): If  $Z_\tau < Z_S$ , then

$$a(t) = \begin{cases} \pi \\ 0 \end{cases} \text{ for } t \in \begin{cases} [\tau, t_S) \\ [t_S, \infty) \end{cases}.$$

The intuition for the zero-investment policy in Case (i) is that the stock level  $Z_\tau$  is "high" ( $Z_\tau \geq Z_S$ ). Then no further investments are needed. (If  $Z_\tau > Z_S$ , the dividend policy is used. If  $Z_\tau = Z_S$ , the singular policy is used). Case (ii) occurs if the stock level  $Z_\tau$  is "low" ( $Z_\tau < Z_S$ ). Investment then is worthwhile, and is continued until the stock  $Z(t)$  reaches the singular level  $Z_S$ .

The policy stated in Proposition 1 (ii) works as follows. Investing at the rate  $a = \pi$  makes the state  $Z$  increase at the maximal rate. The purpose is to get as quickly as possible from the initial level  $Z_\tau$  to the singular level  $Z_S$ . Once the latter is reached, it is optimal to stay there forever. In Case (i) one would also like to approach the singular level as fast as possible. In our setup, where disinvestment is impossible and there is no exogeneous decay of the capital stock,  $Z_S$  clearly cannot be reached if  $Z_\tau > Z_S$ . The best option then is to stay forever at the level  $Z_\tau$ . Clearly, if  $Z_\tau = Z_S$  the firm is already at the singular stock level. This is, however, a hairline case.

For a further interpretation of the results of the proposition, consider Case (i). Since there is no investment, the stock  $Z(t)$  is constant, equal to, say  $\tilde{Z}$ . The elementary probability of the time interval  $[t, t + dt)$  for the

stopping time  $T$  then is

$$\gamma \tilde{Z} e^{-\int_{\tau}^t \gamma \tilde{Z} ds} = \gamma \tilde{Z} e^{-\gamma \tilde{Z}(t-\tau)}.$$

The expected present value of employing a zero-investment policy, as of time  $\tau$ , is given by

$$e^{i\tau+Y_{\tau}} \int_{\tau}^{\infty} \{e^{-[it+Y(t)]} [\pi + \bar{P}\gamma\tilde{Z}] dt - \beta L_0 = \frac{\pi + \bar{P}\gamma\tilde{Z}}{i + \gamma\tilde{Z}} - \beta L_0. \quad (13)$$

Now suppose, hypothetically, that a marginal investment were made. This would increase marginally the stock of IC. Differentiation in (13) with respect to  $\tilde{Z}$  provides the expected marginal revenue of such investment:

$$\frac{\gamma(i\bar{P} - \pi)}{(i + \gamma\tilde{Z})^2}.$$

Equating this expected marginal revenue to the marginal cost of investment, which is one, yields

$$\gamma(i\bar{P} - \pi) - (i + \gamma\tilde{Z})^2 = 0. \quad (14)$$

Using (10), (14), and Proposition 1 we conclude the following. For  $\tilde{Z} > Z_S$ , an investment would make marginal cost exceed expected marginal revenue. This is why no investment should take place. If  $\tilde{Z} = Z_S$ , expected marginal revenue equals marginal cost. This is a singular arc, along which

$a = 0$ . Finally, if  $Z < Z_S$ , expected marginal revenue of investment exceeds marginal cost, at least during some initial interval of time, say,  $[\tau, \tau + \Delta]$ . In that case, a dividend policy is suboptimal and investment is worthwhile. Thus,  $a(t) = \pi$  on the time interval  $(\tau, t_S]$ . Investment is stopped at time  $t_S$ , because at this instant, the stock reaches the singular level  $Z_S$ .

### 3.2. Complete Solution of the Investment Problem

The objective functional of this problem, when cast as a deterministic optimal control problem, is

$$\begin{aligned}
 J(a(\cdot); Z_0) = & \int_0^\tau e^{-[it+Y(t)]} [\pi - a(t) + (1 - \theta(L_0))\bar{P}\gamma Z(t)] dt + \\
 & e^{-i\tau} J_2^*(Z_\tau, Y_\tau).
 \end{aligned} \tag{15}$$

For  $t \in [0, \tau)$ , the objective to maximized is the integral on the right-hand side of (15). The dynamics are

$$\begin{aligned}
 \dot{Z}(t) &= a(t), \quad Z(0) = Z_0 \\
 \dot{Y}(t) &= \gamma Z(t), \quad Y(0) = 0.
 \end{aligned} \tag{16}$$

Denote this subproblem by  $P1$  and note (also here) that one cannot verify sufficiency conditions since the concavity requirements are not fulfilled.

The characterization of an optimal investment policy in problem  $P1$  is

similar to that in  $P2$  and it suffices to note the following. Defining a costate  $\mu_1(t)$  associated with state variable  $Z(t)$ , one can identify three candidate policies:

$$\begin{aligned} \textit{Investment} & : \quad a(t) = \pi \implies \mu_1(t)e^{it+Y(t)} > 1 \\ \textit{Dividend} & : \quad a(t) = 0 \implies \mu_1(t)e^{it+Y(t)} < 1 \\ \textit{Singular} & : \quad a(t) \in [0, \pi] \implies \mu_1(t)e^{it+Y(t)} = 1. \end{aligned}$$

An optimal solution of  $P1$  is characterized in Proposition 2, in which the singular stock level is given by

$$Z_{SA} = \frac{1}{\gamma} \left( -i + \sqrt{\gamma(i(1 - \theta(L_0))\bar{P} - \pi)} \right). \quad (17)$$

To have a positive singular level  $Z_{SA}$ , we strengthen the assumption in (11) by introducing the hypothesis

$$\bar{P} > \frac{i}{\gamma} + \frac{\pi}{i} + \theta(L_0)\bar{P}.$$

Note

$$Z_{SA} < Z_S,$$

that is, the singular stock level in problem  $P1$  than in  $P2$ . The reason clearly is that in problem  $P1$ , the reward  $\bar{P}$  must be shared with the venture capitalist.

If  $Z_0 < Z_{SA}$  the instant of time at which  $Z(t)$  reaches  $Z_{SA}$ , when

starting out from  $Z_0$  and using the investment policy  $a(t) \equiv \pi$ , is given by

$$t_{SA} = \frac{Z_{SA} - Z_0}{\pi}. \quad (18)$$

**Proposition 2** *Assume that  $Z_{SA} - \pi\tau > 0$ . For any  $Z_0 > 0$ , an optimal solution is:*

(i): *If  $Z_0 \geq Z_{SA}$ , then*

$$a(t) = 0 \text{ for } t \in [0, \tau).$$

(ii): *If  $Z_{SA} - \pi\tau \leq Z_0 < Z_{SA}$ , then*

$$a(t) = \begin{cases} \pi \\ 0 \end{cases} \text{ for } t \in \begin{cases} [0, t_{SA}) \\ [t_{SA}, \tau) \end{cases}.$$

(iii): *If  $Z_0 < Z_{SA} - \pi\tau$ , then*

$$a(t) = \pi \text{ for } t \in [0, \tau).$$

The intuition of Case (i) is that it occurs if the initial stock  $Z_0$  is already sufficiently high; in particular, it exceeds the singular level  $Z_{SA}$ . Then it does not pay to invest in order to increase the stock of IC.

In Cases (ii) and (iii), the initial stock  $Z_0$  is insufficient and investments are used to increase it (maximally). In Case (iii), investments continue throughout the time interval  $[0, \tau)$ . In Case (ii), investment is stopped

at time  $t_{SA}$ , i.e., at the instant of time where the singular stock level is reached.

To see the difference between (ii) and (iii) note that

$$t_{SA} \begin{cases} > \\ = \\ < \end{cases} \tau \iff Z_0 \begin{cases} < \\ = \\ > \end{cases} Z_{SA} - \pi\tau. \quad (19)$$

In Case (iii) we have  $t_{SA} > \tau \iff Z_0 < Z_{SA} - \pi\tau$ . This means that even when investment is done at the maximal rate  $\pi$  during the entire time interval  $[0, \tau)$ , it cannot increase the initial stock  $Z_0$  to the singular level  $Z_{SA}$ . In Case (ii) we have  $t_{SA} \leq \tau \iff Z_0 \geq Z_{SA} - \pi\tau$ , which means that investing throughout the time interval  $[0, \tau)$  would overshoot the singular level. Therefore, investment goes on only during the time interval  $[0, t_{SA})$ . Clearly, the initial stock  $Z_0$  is larger in Case (ii) than in Case (iii); hence there is less need for investments.

We are now ready to state a main result of the paper, the characterization of an optimal solution of the full problem. This result is stated as Proposition 3.

**Proposition 3** *For any  $Z_0 > 0$ , an optimal solution is:*

(i): *If  $Z_0 \geq Z_S$ , then*

$$a(t) = 0 \text{ for } t \in [0, \infty).$$

(ii): If  $Z_{SA} \leq Z_0 < Z_S$ , then

$$a(t) = \begin{pmatrix} 0 \\ \pi \\ 0 \end{pmatrix} \quad \text{for } t \in \begin{pmatrix} [0, \tau) \\ [\tau, t_S) \\ [t_S, \infty) \end{pmatrix}.$$

(iii): If  $Z_{SA} - \pi\tau < Z_0 < Z_{SA}$ , then

$$a(t) = \begin{pmatrix} \pi \\ 0 \\ \pi \\ 0 \end{pmatrix} \quad \text{for } t \in \begin{pmatrix} [0, t_{SA}) \\ [t_{SA}, \tau) \\ [\tau, t_S) \\ [t_S, \infty) \end{pmatrix}.$$

(iv): If  $Z_0 \leq Z_{SA} - \pi\tau$ , then

$$a(t) = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \quad \text{for } t \in \begin{pmatrix} [0, t_S) \\ [t_S, \infty) \end{pmatrix}.$$

Proposition 3 is derived from Propositions 1 and 2. The four cases in Proposition 3 are ordered in accordance with increasing investment efforts, or, equivalently, decreasing initial stocks of IC.

In Case (i), the initial stock level  $Z_0$  is sufficiently high and there is no need for investments at all. In Case (ii), the stock level  $Z_0$  is less than the "long term" singular level  $Z_S$  and it pays to invest to build up intellectual capital. Investments are, however, postponed until the exit of the venture



capitalist. The reason is that  $Z_0$  exceeds the "short term" singular stock level  $Z_{SA}$ . In Case (iii) we see a sort of "pulsing strategy" where investment switches twice between maximal effort and zero. Although the long term singular level has not yet been reached at time  $t_{SA}$ , investment is temporarily stopped - until the venture capitalist exits. Then investment is resumed and is continued until the long term singular stock level is reached. In Case (iii), the initial stock is the lowest in the four cases, and it pays to keep on investing until reaching the long term singular level  $Z_S$ .

In Cases (ii), (iii), and (iv) we see the implications of the *knowledge effect*, which reflects the fact that the firm's past efforts contribute to its chances of making the discovery. Due to that effect, the firm scales down its investments in IC as the stock of knowledge (IC) increases (Doraszelski (2003)). In such cases, investment expenditures can be reduced (although rather abruptly) as the stock of IC increases. (In a nonlinear model, one would see a smoother decrease in the investment rate).

#### 4. Financial Policy

The derivations of Section 3 proceeded under the assumption that the amount of venture capital,  $L_0$ , was fixed. It remains to discuss the choice of  $L_0$ . This amount can be seen as a parameter, determined at time zero, that influences the dynamic optimization problem.

We assumed that venture capital can only be used for an initial, one-

shot, investment in the stock of IC. For the stockholders, the intertemporal trade-off is between giving up current consumption in order to invest (to increase the probability of winning the reward  $\bar{P}$ ), and refraining from investment in order to consume now. We have seen that this decision depends on the stock level  $Z_0$ . Recalling that  $Z_0 = X_0 + L_0$  shows that the venture capital decision will influence the investment decision. Thus, increasing the amount of venture capital will increase  $Z_0$ , the initial stock of IC, and hence reduce the need for later investments (see, for example, Proposition 3, Case (i)). On the other hand, increasing the amount of venture capital will increase  $\beta L_0$ , the exit payment to the venture capitalist as well as the capitalist's share,  $\theta(L_0)$ , of the reward.

The occurrence of the four cases in Proposition 3 depend on the amount of venture capital,  $L_0$ , such that Case (i) occurs if  $L_0 \geq Z_S - X_0$ , Case (ii) if  $Z_{SA} - X_0 \leq L_0 \leq Z_S - X_0$ , Case (iii) if  $Z_{SA} - \pi\tau - X_0 < L_0 < Z_{SA} - X_0$ , and Case (iv) if  $L_0 \leq Z_{SA} - \pi\tau - X_0$ . Recall that in Case (i) the firm does not invest at all. The intuition is that the amount of venture capital attracted, and used in the initial acquisition of IC, is sufficiently large. In Cases (ii), (iii), and (iv) there is an initial phase of investment, which is warranted by the fact that in these cases the amount of venture capital is smaller. Investments go on for the longest interval of time in Case (iv), which is intuitive since here the amount of venture capital is the smallest

Denote by  $J(L_0)$  the firm's payoff over the time interval  $[0, \infty)$ . Using (7) and (15) yields

$$J(L_0) = \int_0^\tau e^{-[it+Y(t)]} [\pi - a(t) + (1 - \theta(L_0)) \bar{P} \gamma Z(t)] dt + \quad (20)$$

$$e^{-i\tau} \left[ e^{i\tau+Y_\tau} \int_\tau^\infty e^{-[it+Y(t)]} [\pi - a(t) + \bar{P} \gamma Z(t)] dt - \beta L_0 \right].$$

To determine the amount  $L_0$ , we apply a necessary optimality condition in Léonard and Long (1992, Th. 7.11.1 ). Whenever  $L_0$  is strictly positive it is given by

$$\theta'(L_0) = \frac{1}{K} [\lambda_1(0) - \beta e^{-i\tau}], \quad (21)$$

in which

$$K = \int_0^\tau e^{-[it+Y(t)]} \bar{P} \gamma Z(t) dt > 0.$$

The condition in (21) compares the costate  $\lambda_1(0)$  (the shadow price of  $Z_0$ ) with the present value of the constant  $\beta$ . Recall that if the venture capitalist exits at the predetermined time  $\tau$ , she is paid the amount  $\beta L_0$ . If the shadow price exceeds the (present value of the) marginal payment to be made at time  $\tau$ , venture capital financing should be attracted in accordance with (21). On the other hand, if  $\beta e^{-i\tau} \geq \lambda_1(0)$ , no capital should be attracted.

The choice in (21) depends on the exit time  $\tau$  and the rate of time preference,  $i$ . To see the implications of varying these parameters, we proceed as follows.

*Case A. Time preference rate fixed.*

The choice of  $L_0 = 0$  is more likely for  $\tau$  small (since the condition in (21) then can be satisfied by a larger range of  $\beta$ s). Thus, if the latest exit time of the venture capitalist is in the near future, the founders do not wish to attract venture capital. The policy could also occur for a large  $\tau$ , but then a large value of  $\beta$  is necessary. The intuition here is that although the exit of the capitalist is quite far in the future, the founders do not attract venture capital due to the large amount,  $\beta L_0$ , that must be paid upon exit of the capitalist.

*Case B. Exit time fixed.*

The choice  $L_0 = 0$  is more likely for  $i$  small. Thus, if the founders are far-sighted, they are reluctant to attract venture capital. The policy could also occur for a large value of  $i$ , but then a large value of  $\beta$  is necessary. The intuition here is that although the founders are myopic, they do not attract venture capital due to the large amount that must be paid upon exit of the capitalist.

Suppose now that the function  $\theta(L_0)$  is convex. To be specific, let  $\theta(L_0) = \frac{1}{2}L_0^2$ . Thus, the share that the venture capitalist receives from the reward increases progressively with the amount of venture capital supplied. This could be seen as a means of hedging the capitalist's investment.

Using (21) yields  $L_0 = \frac{\lambda_1(0) - \beta e^{-i\tau}}{K}$ . In this expression, a direct and an indirect effect can be identified. The direct effect is that the amount of venture capital decreases if the reward  $\bar{P}$  increases. The reason is that a

marginal increase in  $L_0$  leads to a larger monetary loss,  $\theta'(L_0)\bar{P}$ , when  $\bar{P}$  is large. The indirect effect concerns the fact that a higher reward increases the incentive to invest in intellectual capital and thus affects  $L_0$  positively. In the model, this is reflected in the fact that the threshold stock  $Z_{SA}$  increases with the reward (cf. (17)). According to Proposition 3 this will make sequences (iii) and (iv) more likely as optimal ones. In these two policies, the shadow price  $\lambda_1(0)$  is sufficiently large to influence positively the amount of venture capital  $L_0$  (cf. (21)).

## 5. Concluding Remarks

The paper has considered a dynamic optimization problem of a firm that tries to make a breakthrough with a new IT product. The probability of achieving a breakthrough can be increased by investing in intellectual capital (IC). The initial investment in IC can be financed by venture capital. The paper has determined optimal investment policies and an optimal amount of venture capital. The singular stock of IC is a main determinant of the type of the investment policy.

We conclude by stating two extensions that should be promising areas for future research.

- Venture capital is usually provided in stages, and not as a lump sum at the start of the venture. This is perhaps the most limiting assumption of the paper. Capital being provided stagewise means that the

firm receives capital as seed money, startup capital, capital for development of prototypes, etc. (see Sahlman (1990, Table 2) for details). A priori, the completion times of the various stages are random, and the state in which a new stage starts out is random, too. The methodology of piecewise deterministic optimal control problems apply here (see, e.g., Carlson et al. (1991), Dockner et al. (2000)).

- Introduce competitors. There is a considerable literature in applied differential games dealing with innovations and R&D in competitive environments (see, e.g., Reinganum (1981, 1982), Dockner et al. (2000)).

## 6. Appendix

### Proof of Proposition 1

In (5), the term  $e^{i\tau+Y_\tau}$  is constant and can be disregarded in the characterization of an optimal solution. Let  $\lambda_1(t), \lambda_2(t)$  be costate variables associated with state variables  $Z$  and  $Y$ , respectively, and  $\eta_1(t), \eta_2(t)$  be Lagrangian multipliers associated with the control constraint  $a(t) \in [0, \pi]$ . The Hamiltonian is

$$H = e^{-(it+Y)}[\pi - a + \bar{P}\gamma Z] + \lambda_1 a + \lambda_2 \gamma Z,$$

and the Lagrangian then becomes

$$L = H + \eta_1 a + \eta_2 (\pi - a).$$

Suppose that there exists an optimal solution of problem *P2*. Necessary optimality conditions then are

$$\lambda_1(t) = \eta_2(t) - \eta_1(t) + e^{-(it+Y(t))} \quad (\text{A.1})$$

$$\dot{\lambda}_1(t) = -e^{-(it+Y(t))} \gamma \bar{P} - \gamma \lambda_2(t) \quad (\text{A.2})$$

$$\dot{\lambda}_2(t) = e^{-(it+Y(t))} [\pi - a(t) + \bar{P} \gamma Z(t)] \quad (\text{A.3})$$

$$\eta_1(t) a(t) = 0, \quad a(t) \geq 0, \quad \eta_1(t) \geq 0$$

$$\eta_2(t) (\pi - a(t)) = 0, \quad a(t) \leq \pi, \quad \eta_2(t) \geq 0$$

$$\lim_{t \rightarrow \infty} \lambda_2(t) = 0. \quad (\text{A.4})$$

Due to the linearity of the Hamiltonian in the control, an optimal investment policy is bang-bang or singular. Clearly, both  $\eta_1$  and  $\eta_2$  cannot be positive and hence three candidate policies remain. Characterized by the signs of the Lagrangian multipliers they are

	<i>Investment</i> ( $a = \pi$ )	<i>Dividend</i> ( $a = 0$ )	<i>Singular</i> ( $a = 0$ )
$\eta_1$	0	+	0
$\eta_2$	+	0	0

The singular investment policy is indeterminate in the interval  $[0, \pi]$  and

we choose  $a = 0$ . The stock level  $Z_S$  associated with this singular policy is constant and given by (10). The instant of time at which the stock  $Z(t)$  reaches the singular level  $Z_S$ , when starting out in state  $Z_\tau$ , is given by

$$t_S = \frac{Z_S - Z_\tau}{\pi}.$$

In what follows we characterize in detail the three candidate policies.

**Investment policy (Path 1)**

$$\eta_1(t) = 0, \eta_2(t) > 0, \dot{Z}(t) = a(t) = \pi.$$

From (A.1) and (A.3) it follows that

$$\lambda_1(t) = e^{-(it+Y(t))} + \eta_2(t) \quad (\text{A.5})$$

$$\dot{\lambda}_2(t) = e^{-(it+Y(t))} \bar{P} \gamma Z(t). \quad (\text{A.6})$$

Differentiating  $\lambda_1(t)$  in (A.5) twice with respect to time yields

$$\dot{\lambda}_1(t) = -e^{-(it+Y(t))} [i + \gamma Z(t)] + \dot{\eta}_2(t) \quad (\text{A.7})$$

$$\ddot{\lambda}_1(t) = e^{-(it+Y(t))} [(i + \gamma Z(t))^2 - \gamma \pi] + \ddot{\eta}_2(t). \quad (\text{A.8})$$

Differentiating in (A.2) with respect to time, and using (A.6) and (A.8), provides

$$\begin{aligned} \ddot{\lambda}_1(t) &= e^{-(it+Y(t))} i \bar{P} \gamma, \\ \ddot{\eta}_2(t) &= e^{-(it+Y(t))} \{ \gamma (i \bar{P} + \pi) - [i + \gamma Z(t)]^2 \} \end{aligned} \quad (\text{A.9})$$



**Dividend policy (Path 2)**

$$\eta_2(t) = 0, \eta_1(t) > 0, \dot{Z}(t) = a(t) = 0, Z(t) = Z_D = \text{const.} > 0.$$

From (A.1) and (A.3) it follows that

$$\lambda_1(t) = e^{-(it+Y(t))} - \eta_1(t) \quad (\text{A.10})$$

$$\dot{\lambda}_2(t) = e^{-(it+Y(t))}[\pi + \bar{P}\gamma Z_D]. \quad (\text{A.11})$$

Differentiating  $\lambda_1(t)$  in (A.10) twice with respect to time yields

$$\dot{\lambda}_1(t) = -e^{-(it+Y(t))}[i + \gamma Z_D] - \dot{\eta}_1(t) \quad (\text{A.12})$$

$$\ddot{\lambda}_1(t) = e^{-(it+Y(t))}[i + \gamma Z_D]^2 - \ddot{\eta}_1(t). \quad (\text{A.13})$$

Differentiating in (A.2) with respect to time and using (A.11) yields

$$\ddot{\lambda}_1(t) = e^{-(it+Y(t))}\gamma(i\bar{P} - \pi)$$

and using (A.13) then provides

$$e^{-(it+Y(t))}[i + \gamma Z_D]^2 - \ddot{\eta}_1(t) = e^{-(it+Y(t))}\gamma(i\bar{P} - \pi) \implies \quad (\text{A.14})$$

$$\ddot{\eta}_1(t) = e^{-(it+Y(t))}[(i + \gamma Z_D)^2 - \gamma(i\bar{P} - \pi)].$$

**Singular policy (Path 3)**

$$\eta_1(t) = \eta_2(t) = 0, \dot{Z}(t) = a(t) = 0, Z(t) = Z_S = \text{const.} > 0.$$

The characterizations of costates and Lagrangian multipliers follow from those of Path 2, by putting  $\eta_1(t)$  equal to zero. Thus

$$\lambda_1(t) = e^{-(it+Y(t))} \quad (\text{A.15})$$

$$\dot{\lambda}_2(t) = e^{-(it+Y(t))} [\pi + \bar{P}\gamma Z_S] \quad (\text{A.16})$$

$$\dot{\lambda}_1(t) = -e^{-(it+Y(t))} (i + \gamma Z_S) \quad (\text{A.17})$$

$$\gamma(i\bar{P} - \pi) - (i + \gamma Z_S)^2 = 0 \iff Z_S = \frac{1}{\gamma} [-i + \sqrt{\gamma(i\bar{P} - \pi)}]. \quad (\text{A.18})$$

We construct a solution that satisfies the necessary optimality conditions, by using a path-coupling procedure which starts by considering Path 2 or Path 3 as a candidate for being a final path. A final path is one that is employed as of some instant of time till infinity. Path 1 will not be considered a candidate for being a final path from the following reason: since the probability of making the innovation is concavely increasing in the stock of IC, it is clearly suboptimal to keep on investing forever.

We start by considering the case where Path 2 is the final path.

**Lemma 4** *If Path 2 is the final path, it cannot be preceded by any other path.*

**Proof.** By contradiction. Since  $Z$  is a "good stock", we can safely assume that the costate  $\lambda_1(t)$  is always non-negative. Note that (A.10)

implies

$$\lim_{t \rightarrow \infty} \lambda_1(t) = 0 \quad (\text{A.19})$$

$$\lim_{t \rightarrow \infty} \eta_1(t) = 0 \quad (\text{A.20})$$

along Path 2.

Now suppose that Path  $k$  ( $k = 1$  or  $3$ ) were to precede Path 2. Then, at the coupling point  $\bar{t}$  between Path  $k$  and Path 2, it must be true that

$$\eta_1(\bar{t}) = 0,$$

which follows from the required continuity of  $\lambda_1(t)$ , and using (A.5), (A.15). By definition,  $\eta_1(t)$  is positive on Path 2 for  $t \in (\bar{t}, \infty)$ . Then, using  $\eta_1(\bar{t}) = 0$  and (A.20), it follows that  $\eta_1(t)$  must have at least one maximum along Path 2. In such a point we have  $\ddot{\eta}_1(t) < 0$ . From (A.14) and the fact that  $Z(t) = \text{const.} = Z_D$  along Path 2, one therefore obtains

$$\gamma(iP - \pi) - (i + \gamma Z_D)^2 > 0$$

everywhere along Path 2. Hence, by (A.14) it holds that  $\ddot{\eta}_1(t) < 0$  everywhere on Path 2. This means that  $\dot{\eta}_1(t)$  decreases along Path 2 (or, equivalently, that  $\eta_1$  is a strictly concave function).

On the other hand, using the necessary conditions (A.2), (A.4), and

(A.12) yields

$$\begin{aligned}\dot{\eta}_1(t) &= e^{-(it+Y(t))}[\bar{P}\gamma - (i + \gamma Z_D)] + \gamma\lambda_2(t) \\ \implies \lim_{t \rightarrow \infty} \dot{\eta}_1(t) &= 0.\end{aligned}\tag{A.21}$$

However, (A.21) cannot hold if  $\dot{\eta}_1$  is decreasing everywhere on Path 2. Thus, supposing that another path can precede Path 2 leads to a violation of the necessary optimality conditions. This observation completes the proof. ■

**Lemma 5** *When Path 2 is used for  $t \in (\tau, \infty)$  it holds that*

$$\gamma(i\bar{P} - \pi) - (i + \gamma Z_D)^2 < 0.$$

**Proof.** Since  $Z$  is constant along Path 2, (A.14) shows that  $\ddot{\eta}_1$  does not change its sign. From this fact, and using (A.20) and (A.21) it follows that  $\ddot{\eta}_1(t) > 0$  everywhere along Path 2, which via (A.14) leads to the result. ■

Next, suppose that Path 3 is the final path, in which case (A.18) is satisfied everywhere along Path 3. For this situation we have the following result.

**Lemma 6** *If Path 3 is the final path, a solution is given by the sequence Path 1  $\rightarrow$  Path 3, where on Path 1 it holds that*

$$\gamma(i\bar{P} - \pi) - (i + \gamma Z(t))^2 > 0.$$

**Proof.** From (A.9) and (A.18) one obtains  $\ddot{\eta}_2(t_{13}^-) > 0$ , where  $t_{13}$  is the moment of time at which Path 1 passes into Path 3. Since  $Z_1(t)$  increases on Path 1, it follows that

$$\ddot{\eta}_2(t) > 0$$

everywhere along Path 1.

From (A.5), (A.15), continuity of  $\lambda_1(t)$ , and the fact that  $\eta_2(t) > 0$  along Path 1, it follows that

$$\eta_2(t_{13}^-) = 0 \tag{A.22}$$

$$\dot{\eta}_2(t_{13}^-) \leq 0. \tag{A.23}$$

Now,  $\ddot{\eta}_2(t) > 0$  on Path 1, (A.22) and (A.23) imply that  $\eta_2(t)$  decreases everywhere along Path 1.

Suppose that another path were to precede Path 1. This would require that at the coupling point,  $\tilde{t}$ , say, between these two paths it must hold that  $\eta_2(\tilde{t}) = 0$  since otherwise continuity  $\lambda_1(t)$  would be violated. However, this is incompatible with (A.22) and the fact that  $\eta_2(t)$  is decreasing everywhere along Path 1. This demonstrates that no path can be coupled before the sequence Path 1  $\rightarrow$  Path 3. The inequality stated in the lemma follows directly follows from (A.18) and the fact that  $Z(t)$  increases on Path 1. ■

**Lemma 7** *Path 2 cannot precede Path 3 if the latter is a final path.*

**Proof.** By contradiction. All along the sequence Path 2  $\rightarrow$  Path 3,

(A.18) holds since  $Z(t)$  is constant on both paths. This implies, using (A.14), that  $\ddot{\eta}_1(t) = 0$  along Path 2 and hence  $\dot{\eta}_1(t)$  is constant.

Now, continuity of  $\lambda_1(t)$  requires that just before a coupling point  $t_{23}$  between Paths 2 and 3 it must hold that  $\eta_1(t_{23}^-) = 0$ , cf. (A.10) and (A.15). Hence  $\dot{\eta}_1(t_{23}^-)$  must be negative, and since  $\dot{\eta}_1$  is constant, it is negative everywhere along Path 2.

From (A.2) follows that  $\dot{\lambda}_1(t)$  is continuous on every path. Using (A.12) and (A.17) shows that this is violated for a strictly negative  $\dot{\eta}_1$ . This completes the proof. ■

We summarize the results of the above lemmas as follows. Consider problem *P2*. Sequences of paths that satisfy the necessary optimality conditions are the following:

$$\begin{aligned} \gamma(i\bar{P} - \pi) - (i + \gamma Z_\tau)^2 < 0 &\iff Z_\tau > Z_S : \text{Path 2} \\ \gamma(i\bar{P} - \pi) - (i + \gamma Z_\tau)^2 = 0 &\iff Z_\tau = Z_S : \text{Path 3} \\ \gamma(i\bar{P} - \pi) - (i + \gamma Z_\tau)^2 > 0 &\iff Z_\tau < Z_S : \text{Path 1} \rightarrow \text{Path 3}. \end{aligned}$$

The two first sequences generate the control path  $a(t) = 0$  for  $t \in [\tau, \infty)$ . The last one generates the control path  $a(t) = \pi$  for  $t \in [\tau, t_S)$ ,  $a(t) = 0$  for  $t \in [t_S, \infty)$ . This completes the proof of Proposition 1.

### Proof of Proposition 2

Essentially, a proof of this proposition is superfluous; in most respects it would be similar to that of Proposition 1.

Proposition 1 deals with an infinite horizon problem with a fixed initial state  $Z_\tau$ . In this problem there is a singular stock level  $Z_S$  and the optimal policy is to reach this level as quickly as possible. If  $Z_\tau < Z_S$ , the firm invests maximally to reach the singular stock level at time  $t_S$ ; due to the infinite horizon,  $Z_S$  will always be reached. On the other hand, if the initial stock  $Z_\tau$  exceeds the singular level  $Z_S$ , no investment is needed.

The situation in Proposition 2 is basically the same. Proposition 2 deals with a finite horizon problem with a fixed initial state  $Z_0$ . In this problem there is also a singular stock level,  $Z_{SA}$ , and the optimal policy is to reach this level as quickly as possible, by using maximal investment efforts. If  $Z_0 < Z_{SA}$ , the firm invests maximally to try to reach the singular stock level at time  $t_{SA}$ . However, due to the finite horizon,  $Z_{SA}$  may not be reached before the end of the horizon,  $\tau$ . Thus, one needs to distinguish Cases (ii) and (iii) in Proposition 2. In Case (ii) there is time enough to reach the singular level, and investment is discontinued during the remaining time interval  $[t_{SA}, \tau)$ . In Case (iii), investment goes on until the end of the horizon, because  $t_{SA} > \tau$ . Finally, as in Proposition 1, if the initial stock  $Z_0$  exceeds the singular level  $Z_{SA}$ , no investment is needed.

### **Proof of Proposition 3**

To establish Proposition 3, we introduce the two functions

$$\begin{aligned} f(Z) &= \gamma(i\bar{P} - \pi) - (i + \gamma Z(t))^2 \\ f_A(Z) &= \gamma(i(1 - \theta)\bar{P} - \pi) - (i + \gamma Z(t))^2, \end{aligned}$$

which both are decreasing and strictly concave. Note that

$$\begin{aligned} f(Z) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 &\iff Z(t) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} Z_S \\ f_A(Z) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 &\iff Z(t) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} Z_{SA}, \end{aligned}$$

and recall that  $Z_{SA} < Z_S$ . It is convenient to use this notation to summarize the results of Propositions 1 and 2:

*Proposition 1:*

- (i): If  $f(Z_\tau) \leq 0$ , then  $a(t) = 0$  for  $t \in [\tau, \infty)$ .
- (ii): If  $f(Z_\tau) > 0$ , then  $a(t) = \pi$  for  $t \in [\tau, t_S)$ ,  $a(t) = 0$  for  $t \in [t_S, \infty)$ .

*Proposition 2:* Let  $Z_{\pi\tau} \triangleq Z_0 + \pi\tau$ .

- (i): If  $f_A(Z_0) \leq 0$ , then  $a(t) = 0$  for  $t \in [0, \tau)$ .
- (ii): If  $f_A(Z_0) > 0$  and  $Z_{\pi\tau} > Z_{SA}$ , then  $a(t) = \pi$  for  $t \in [0, t_{SA})$ ,  $a(t) = 0$  for  $t \in [t_{SA}, \tau)$ .
- (iii): If  $f_A(Z_0) > 0$  and  $Z_{\pi\tau} \leq Z_{SA}$ , then  $a(t) = \pi$  for  $t \in [0, \tau)$ .

Now we can prove Proposition 3.

**Case (i):**

The assumption is  $f(Z_0) \leq 0$ , which implies  $f_A(Z_0) < 0$ . Hence, by Proposition 2(i), we have  $a(t) = 0$  for  $t \in [0, \tau)$ , which implies  $Z_\tau = Z_0$ .



Then, by the assumption of Case (i), one obtains  $f(Z_\tau) = f(Z_0) \leq 0$ . Using Proposition 1(i) then yields  $a(t) = 0$  for  $t \in [\tau, \infty)$ . The proof is complete.

**Case (ii):**

The assumption is  $f_A(Z_0) \leq 0$  and  $f(Z_0) > 0$ . The first inequality implies, by Proposition 2(i), that  $a(t) = 0$  for  $t \in [0, \tau)$ . Then one has  $Z_\tau = Z_0$ , which implies  $f(Z_\tau) = f(Z_0) > 0$ . Using Proposition 1(ii) then yields  $a(t) = \pi$  for  $t \in [\tau, t_S)$  and  $a(t) = 0$  for  $t \in [t_S, \infty)$ . The proof is complete.

**Case (iii):**

The assumption is  $f_A(Z_0) > 0$  and  $Z_0 > Z_{SA} - \pi\tau$ . The first inequality implies, by Proposition 2(ii), that  $a(t) = \pi$  for  $t \in [0, t_{SA})$  and  $a(t) = 0$  for  $t \in [t_{SA}, \tau)$ . Then  $Z_\tau = Z_{SA}$  in Proposition 1 and hence  $f(Z_\tau) = f(Z_{SA})$ . However,  $f(Z_{SA}) > 0$ , and using Proposition 1(ii) one obtains  $a(t) = \pi$  for  $t \in [\tau, t_S)$  and  $a(t) = 0$  for  $t \in [t_S, \infty)$ . The proof is complete.

**Case (iv):**

The assumption is  $f_A(Z_0) > 0$ . By Proposition 2(iii) we get  $a(t) = \pi$  for  $t \in [0, \tau)$ . This implies  $Z_\tau < Z_{SA} < Z_S$ , and hence  $f(Z_\tau) > f(Z_S) = 0$ . Then, by Proposition 1(ii), we get  $a(t) = \pi$  for  $t \in [\tau, t_S)$  and  $a(t) = 0$  for  $t \in [t_S, \infty)$ . The proof is complete.

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